From Laurent Series to Exact Meromorphic Solutions: the Kawahara equation

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Abstract

Nonlinear waves are studied in a mixture of liquid and gas bubbles. Influence of viscosity and heat transfer is taken into consideration on propagation of the pressure waves. Nonlinear evolution equations of the second and the third order for describing nonlinear waves in gas-liquid mixtures are derived. Exact solutions of these nonlinear evolution equations are found. Properties of nonlinear waves in a liquid with gas bubbles are discussed.

1   Introduction

In this article we study meromorphic traveling wave solutions of the following partial differential equations

$$u_t + \alpha u^n u_x + \beta u_{xxx} - \delta u_{xxxxx} = 0, \quad \alpha \neq 0, \quad \delta \neq 0, \tag{1}$$

where $n = 1$, $n = 2$, and $n = 4$. In the case $n = 1$ equation (1) is the famous Kawahara equation, arising in several physical applications. For example, in the theory of magneto–acoustic waves in plasma [1] and in the theory of long waves in shallow liquid under ice cover [2]. In the case $n = 2$ and $n = 4$ equation (1) may be regarded as modifications of the Kawahara equation. Let us call these equations the modified Kawahara equations. Several families of exact solutions to the Kawahara equation and the modified Kawahara equation (1) with $n = 2$ are given in [3–9].

The problem of constructing exact solutions for nonlinear differential equations is intensively studied in recent years [5,6,10–17]. A lot of methods

and new families of exact solutions appear [18–20]. However, the question of classification for exact solutions is addressed very seldom (see [21–25]). In this article we describe a method, which can be used to classify and build in explicit form meromorphic solutions of autonomous nonlinear ordinary differential equations.

This article is organized as follows. In section 2 we give explicit expressions for meromorphic solutions and describe a process of their construction. In sections 3, 4, and 5 we find all meromorphic solutions of third order ordinary differential equations satisfied by traveling wave solutions of the Kawahara equation and the modified Kawahara equations.

2   Method applied

Equation (1) admits the traveling wave reduction $u(x,t) = w(z)$, $z = x - C_0 t$ with $w(z)$ satisfying the equation

$$\delta w_{zzzz} - \beta w_{zz} - \frac{\alpha}{n+1} w^{n+1} + C_0 w + C_1 = 0, \tag{2}$$

where C_1 is an integration constant. Multiplying equation (2) by w_z and integrating the result, we obtain the equation

$$\delta \left(w_{zzz} w_z - \frac{1}{2} w_{zz}^2 \right) - \frac{\beta}{2} w_z^2 - \frac{\alpha}{(n+1)(n+2)} w^{n+2} + \frac{C_0}{2} w^2 + C_1 w + C_2 = 0, \tag{3}$$

where again C_2 is an integration constant. Without loss of generality, the parameters α, δ in (1) may be taken arbitrary. Let us set

$$
\begin{aligned}
n = 1: &\quad \delta = 1, \quad \alpha = 6; \\
n = 2: &\quad \delta = 1, \quad \alpha = 360; \\
n = 4: &\quad \delta = 1, \quad \alpha = 120.
\end{aligned}
\tag{4}
$$

The aim of this article is to obtain all the families of meromorphic solutions for equation (3) with $n = 1$, $n = 2$, $n = 4$. We shall use an approach suggested in [25]. Consider an autonomous nonlinear ordinary differential equation

$$E[w(z)] = 0, \tag{5}$$

where $E[w(z)]$ is a polynomial in $w(z)$ and its derivatives such that substituting $w(z) = \lambda W(z)$ into equation (5) yields an expression with only one term of the highest degree in respect of λ. For every solution $w(z)$ of equation

(5) there exists a family of solutions $w(z - z_0)$. We shall omit arbitrary constant z_0. Suppose that equation (5) possesses only one asymptotic expansion corresponding to the Laurent series in a neighborhood of the pole $z = 0$

$$w(z) = \sum_{k=1}^{p} \frac{c_{-k}}{z^k} + \sum_{k=0}^{\infty} c_k z^k, \quad 0 < |z| < \varepsilon, \tag{6}$$

Here $p > 0$ is an order of the pole $z = 0$. For example, this is the case of equation (3) with $n = 1$. Exact meromorphic solutions of equation (5) satisfied by one formal Laurent expansion (6) are given in theorem 1 [25].

Theorem 1. All meromorphic solutions of equation (5) with only one asymptotic expansion (6) corresponding to the Laurent series in a neighborhood of the pole $z = 0$ are of the form:

1) elliptic solutions with the periods $2\omega_1$, $2\omega_2$ and one pole of order p inside a parallelogram of periods

$$w(z) = \left\{ \sum_{k=2}^{p} \frac{(-1)^k c_{-k}}{(k-1)!} \frac{d^{k-2}}{dz^{k-2}} \right\} \wp(z; \omega_1, \omega_2) + h_0, \tag{7}$$

Necessary condition for elliptic solutions to exist is $c_{-1} = 0$.
2) simply periodic solutions with the period T and one pole of order p inside a stripe of periods built on T

$$w(z) = \frac{\pi}{T} \left\{ \sum_{k=1}^{p} \frac{(-1)^{k-1} c_{-k}}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \right\} \cot\left(\frac{\pi z}{T} \right) + h_0. \tag{8}$$

3) rational solutions

$$w(z) = \sum_{k=1}^{p} \frac{c_{-k}}{z^k} + \sum_{k=0}^{m} c_k z^k, \quad m \geq 0,$$

$$\tag{9}$$

$$w(z) = \sum_{k=0}^{m} h_k z^k, \quad m \geq 0.$$

In 1), 2), and 3) h_k, $0 \leq k \leq m$ are constants and the Weierstrass elliptic function $\wp$ satisfies the equation

$$(\wp_z)^2 = 4\wp^3 - g_2 \wp - g_3. \tag{10}$$

Now suppose that equation (5) possesses N different asymptotic expansions corresponding to the Laurent series in a neighborhood of the pole $z = 0$

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$$w^{(i)}(z) = \sum_{k=1}^{p_i} \frac{c_{-k}^{(i)}}{z^k} + \sum_{k=0}^{\infty} c_k^{(i)} z^k, \quad 0 < |z| < \varepsilon_i, \quad i = 1, \ldots, N. \qquad (11)$$

In this expressions $p_i > 0$ is an order of the pole $z = 0$. For instance, equation (3) with $n = 2$ admits two different formal Laurent series and equation (3) with $n = 4$ admits four different formal Laurent series. Let us call any pole $z = b$ of meromorphic solution $w(z)$ with the Laurent expansion $w^{(i)}(z - b)$ (see (11)) as a pole of type i. Meromorphic solutions of equation (5) are classified in theorem 1 [25]. Again we omit arbitrary constant z_0.

Theorem 2. All meromorphic solutions of equation (5) with N different asymptotic expansions (11) corresponding to the Laurent series in a neighborhood of the pole $z = 0$ are of the form:

1) elliptic solutions with the periods $2\omega_1$, $2\omega_2$ and $|I| + 1$ poles $\{a_i\}$ of orders $\{\mathrm{ord}\, a_i = p_i\}$, $i \in I \cup \{i_0\}$ inside a parallelogram of periods

$$w(z) = \left\{ \sum_{i \in I} \sum_{k=2}^{p_i} \frac{(-1)^k c_{-k}^{(i)}}{(k-1)!} \frac{d^{k-2}}{dz^{k-2}} \right\} \left(\frac{1}{4} \left[\frac{\wp_z(z) + B_i}{\wp(z) - A_i} \right]^2 - \wp(z) \right)$$

(12)

$$+ \sum_{i \in I} \frac{c_{-1}^{(i)}(\wp_z(z) + B_i)}{2(\wp(z) - A_i)} + \left\{ \sum_{k=2}^{p_{i_0}} \frac{(-1)^k c_{-k}^{(i_0)}}{(k-1)!} \frac{d^{k-2}}{dz^{k-2}} \right\} \wp(z) + h_0,$$

where $\wp(z) \overset{def}{=} \wp(z; \omega_1, \omega_2)$, $a_{i_0} = 0$, $A_i \overset{def}{=} \wp(a_i)$, $B_i \overset{def}{=} \wp_z(a_i)$, $B_i^2 = 4A_i^3 - g_2 A_i - g_3$, $i \in I$. Necessary condition for elliptic solutions (12) to exist is

$$\sum_{i \in I} c_{-1}^{(i)} + c_{-1}^{(i_0)} = 0. \qquad (13)$$

2) simply periodic solutions with the period T and $|I| + 1$ poles $\{a_i\}$ of orders $\{\mathrm{ord}\, a_i = p_i\}$, $i \in I \cup \{i_0\}$ inside a stripe of periods build on T

$$w(z) = \sqrt{L} \left\{ \sum_{i \in I} \sum_{k=1}^{p_i} \frac{(-1)^{k-1} c_{-k}^{(i)}}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \right\} \frac{A_i \cot\left(\sqrt{L} z\right) + \sqrt{L}}{A_i - \sqrt{L} \cot\left(\sqrt{L} z\right)}$$

(14)

$$+ \sqrt{L} \left\{ \sum_{k=1}^{p_{i_0}} \frac{(-1)^{k-1} c_{-k}^{(i_0)}}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \right\} \cot\left(\sqrt{L} z\right) + h_0,$$

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where $a_{i_0} = 0$, $L \stackrel{def}{=} \pi^2/T^2$, $A_i \stackrel{def}{=} \sqrt{L}\cot(\sqrt{L}a_i)$.

3) rational solutions

$$w(z) = \sum_{k=1}^{p_{i_0}} \frac{c_{-k}^{(i_0)}}{z^k} + \sum_{i \in I} \sum_{k=1}^{p_i} \frac{c_{-k}^{(i)}}{(z-a_i)^k} + \sum_{k=0}^{m} h_k z^k, \quad m \geq 0$$

$$(15)$$

$$w(z) = \sum_{k=0}^{m} h_k z^k, \quad m \geq 0.$$

In 1), 2) and 3) h_k, $0 \leq k \leq m$ are constants and $I = \varnothing$ or $I \subseteq \{1, 2, \ldots, N\}$ $\setminus \{i_0\}$, $1 \leq i_0 \leq N$.

The proof of these theorems is based on the Mittag–Leffler's expansions for meromorphic functions (see [25]). Necessary condition (13) for existence of elliptic solutions follows from the theorem for total sum of the residues at poles inside a parallelogram of periods of an elliptic function. Invariants g_2, g_3 such that $g_2^3 - 27g_3^2 \neq 0$ uniquely determine the elliptic function $\wp(z)$. In addition we have the following correlations

$$g_3 = \sum_{(n, m) \neq (0, 0)} \frac{60}{(2n\omega_1 - 2m\omega_2)^4}, \quad g_2 = \sum_{(n, m) \neq (0, 0)} \frac{140}{(2n\omega_1 - 2m\omega_2)^6}. \quad (16)$$

In the case $g_2^3 - 27g_3^2 = 0$ the elliptic function $\wp(z)$ degenerates and consequently elliptic solutions (12) degenerate. Note that for fixed values of parameters in equation (5), if any, there may exist only one meromorphic solution (rational, simply periodic or elliptic) with a pole at $z = 0$ of type i, $1 \leq i \leq N$.

Thus we see that the problem of constructing exact meromorphic solutions of equation (5) in explicit form reduces to the question of finding coefficients for solutions given in theorems 1 and 2. The basic idea is to expand these exact solutions in a neighborhood of their poles and to compare coefficients of these expansions with coefficients of series (11).

At *the first step* for solutions of equation (5) one constructs asymptotic expansions corresponding to Laurent series in a neighborhood of the pole $z = 0$.

At *the second step* one selects an expression for a meromorphic solution $w(z)$(see theorem 2). If the meromorphic solution $w(z)$ possesses poles of types $i \in J$, $J \subseteq \{1, 2, \ldots, N\}$, then one may take any $i \in J$ in capacity of i_0 and suppose that the point $z = 0$ is the pole of type i_0 for $w(z)$.

At *the third step* one expands the meromorphic solution $w(z)$ in a neighborhood of the poles $\{a_i\}$, $i \in I \cup \{i_0\}$, $a_{i_0} = 0$. For elliptic solutions with

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the poles $\{a_i\}$ of orders $\{\text{ord } a_i = p_i\}$, $i \in I \cup \{i_0\}$ inside a parallelogram of periods one takes the expression

$$w(z) = \sum_{i \in I \cup \{i_0\}} c_{-1}^{(i)} \zeta(z - a_i) + \left\{ \sum_{i \in I \cup \{i_0\}} \sum_{k=2}^{p_i} \frac{(-1)^k c_{-k}^{(i)}}{(k-1)!} \frac{d^{k-2}}{dz^{k-2}} \right\} \wp(z - a_i) + \tilde{h}_0,$$

(17)

where $\tilde{h}_0$ is a constant, $a_{i_0} = 0$, and $\zeta(z)$ is the Weierstrass ζ–function, finds the Laurent series for $w(z)$ given by (17) around its poles $z = 0$, $z = a_i$, $i \in I$. And then introduces notation $A_i \stackrel{def}{=} \wp(a_i)$, $B_i \stackrel{def}{=} \wp_z(a_i)$, $i \in I$, $h_0 \stackrel{def}{=} \tilde{h}_0 - \sum_{i \in I} c_{-1}^{(i)} \zeta(a_i) - \sum_{i \in I} c_{-2}^{(i)} \wp(a_i)$. Expression (17) can be rewritten as (12) with the help of addition formulae for the functions $\wp$ and ζ. For simply periodic solutions with the poles $\{a_i\}$ of orders $\{\text{ord } a_i = p_i\}$, $i \in I \cup \{i_0\}$ inside a stripe of periods one takes the expression

$$w(z) = \frac{\pi}{T} \left\{ \sum_{i \in I \cup \{i_0\}} \sum_{k=1}^{p_i} \frac{(-1)^{k-1} c_{-k}^{(i)}}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \right\} \cot\left(\frac{\pi(z - a_i)}{T} \right) + h_0, \quad (18)$$

where $a_{i_0} = 0$, expands this function in a neighborhood of its poles $z = 0$, $z = a_i$, $i \in I$, and introduces notation $L \stackrel{def}{=} \pi^2/T^2$, $A_i \stackrel{def}{=} \sqrt{L} \cot(\sqrt{L} a_i)$. Expression (18) can be rewritten as (14). Note that it may be taken any value for square root of L, for example, with sign $+$.

At *the fourth step* one compares coefficients of the series found at the first and the third steps and solves an algebraic system for the parameters of exact meromorphic solution. In addition correlations for the parameters of equation (5) may arise. Optimal number of equations in the algebraic system equals to the number of parameters of the meromorphic solution and equation (5) plus 1. If this system is inconsistent, then equation (5) does not possess meromorphic solutions with supposed expression. The parameters of exact meromorphic solution (12) that one needs to calculate are g_2, g_3, h_0, A_i, B_i $i \in I$ in the case of elliptic solutions, h_0, L, A_i $i \in I$ in the case of simply periodic solutions, and a_i, $i \in I$, $\{h_k\}$ in the case of rational solutions. In order to find the highest exponent m (see (15)) one may construct the Laurent expansion in a neighborhood of infinity for $w(z)$.

At *the fifth step* one verifies the meromorphic solution obtained at the previous step, substituting this solution and correlations on the parameters of equation (5) into the latter.

In order to construct all the meromorphic solutions for equation (5) one should consider all the variants for possible types of poles possesses by $w(z)$,

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i.e. all subsets of $\{1, 2, \ldots, N\}$. Finally, we would like to mention that our approach may be applied to build exact meromorphic solutions of autonomous nonlinear ordinary differential equations with arbitrary constants in Laurent series in a neighborhood of poles. However these equations may possess meromorphic solutions of more complicated structure.

3   Exact solutions of the Kawahara equation

In this section we find all cases when the following third order ordinary differential equation

$$w_{zzz}w_z - \frac{1}{2}w_{zz}^2 - \frac{\beta}{2}w_z^2 - w^3 + \frac{C_0}{2}w^2 + C_1 w + C_2 = 0, \tag{19}$$

possesses meromorphic solutions and these solutions themselves. Equation (19) arises as traveling wave reduction of the Kawahara equation (equation (1) with $n = 1$). Again we omit arbitrary constant z_0. Equation (19) admits one formal Laurent series in a neighborhood of the pole $z = 0$. This series is the following

$$w(z) = \frac{280}{z^4} - \frac{140\beta}{39z^2} + \frac{507C_0 - 31\beta^2}{3042} - \frac{31\beta^3 z^2}{237276} + \ldots \tag{20}$$

The Fuchs indices for this expansion are -1, $(11 \pm \sqrt{159}i)/2$. Consequently, there are no arbitrary constants in (20). General expression for an elliptic solution (see (7)) can be written as

$$w(z) = \frac{c_{-4}}{6}\wp_{zz} + c_{-2}\wp + h_0, \quad c_{-4} = 280, \quad c_{-2} = -\frac{140\beta}{39}. \tag{21}$$

Necessary condition $c_{-1} = 0$ is automatically satisfied. Finding the Laurent series for this function in a neighborhood of the point $z = 0$, we get

$$w(z) = \frac{c_{-4}}{z^4} + \frac{c_{-2}}{z^2} + \frac{c_{-4}g_2}{60} + h_0 + \left(\frac{c_{-4}g_3}{14} + \frac{c_{-2}g_2}{20}\right)z^2 + o\left(|z|^2\right), \ 0 < |z| < \varepsilon_1. \tag{22}$$

Comparing coefficients of this series with coefficients of expansion (20), we find the parameters h_0, g_3

$$h_0 = \frac{C_0}{6} - \frac{14\,g_2}{3} - \frac{31\beta^2}{3042}, \quad g_3 = \frac{7\beta\,g_2}{780} - \frac{31\beta^3}{4745520} \tag{23}$$

and the correlation on the parameters of equation (19)

$$C_2 = \left(\frac{287}{207} C_0^2 + \frac{1148}{69} C_1 - \frac{41328}{656903} \beta^4 \right) g_2 + \frac{372}{4826809} \beta^6$$

(24)

$$+ \frac{5}{18252} C_0^2 \beta^2 + \frac{5}{1521} \beta^2 C_1 - \frac{C_0^3}{108} - \frac{C_0 C_1}{6}.$$

In expressions (23), (24) g_2 is a solutions of quadratic equation

$$g_2^2 - \frac{\beta^2}{1014} g_2 + \frac{1457\beta^4}{662158224} - \frac{C_0^2}{23184} - \frac{C_1}{1932} = 0.$$

(25)

Simplifying (21), we obtain elliptic solutions of equation (19)

$$w(z) = 280\wp^2 - \frac{140\beta}{39}\wp + \frac{C_0}{6} - \frac{31}{3042}\beta^2 - 28\delta\, g_2.$$

(26)

Substituting this expression into equation (19), we see that equation (19) indeed possesses solutions of the form (26) provided that correlations (24), (25) hold.

Now let us construct simply periodic solutions. Expression (8) with $p = 4$ yields

$$w(z) = -\sqrt{L} \left\{ \frac{c_{-4}}{6} \frac{d^3}{dz^3} - c_{-2} \frac{d}{dz} \right\} \cot(\sqrt{L}z) + h_0, \quad \sqrt{L} = \frac{\pi}{T}.$$

(27)

Expanding this function around the point $z = 0$, we obtain

$$w(z) = \frac{c_{-4}}{z^4} + \frac{c_{-2}}{z^2} + \frac{c_{-4}L^2}{45} + \frac{c_{-2}L}{3} + h_0 + \frac{(20c_{-4}L + 63c_{-2})L^2}{945} z^2$$

(28)

$$+ o\left(|z|^2\right), \quad 0 < |z| < \varepsilon_2.$$

Comparing coefficients of the series (28) and (20), we find the parameters L, h_0 and correlations on the parameters of equation (19). Verification shows that equation (19) possesses solutions of the form

$$w(z) = 280 L_j^2 \cot^4\left(\sqrt{L_j}z\right) + \frac{(140L_j - \beta)L_j}{39} \cot^2\left(\sqrt{L_j}z\right) + \frac{784L_j^2}{9}$$

(29)

$$+ \frac{C_0}{6} - \frac{31\beta^2}{3042} - \frac{280L_j\beta}{117}, \quad L_j = \frac{\beta\kappa_j}{52}, \quad j = 1, 2, 3.$$

if the following correlations are valid

$$C_1 = \left(\frac{38801}{34273200} \kappa^2 - \frac{4991}{3427320} \kappa + \frac{40889}{34273200} \right) \beta^4 - \frac{C_0{}^2}{12},$$

$$C_2 = \left(\frac{11113903}{193072360000} - \frac{951419}{579217080000} \kappa^2 - \frac{1944971}{57921708000} \kappa \right) \beta^6 \quad (30)$$

$$+ \left(\frac{4991}{20563920} \kappa - \frac{38801}{205639200} \kappa^2 - \frac{40889}{205639200} \right) C_0 \beta^4 + \frac{C_0{}^3}{216}.$$

Three values for the parameter κ are the following

$$\kappa_1 = -1, \quad \kappa_2 = \frac{31 + 3\sqrt{31}i}{20}, \quad \kappa_3 = \frac{31 - 3\sqrt{31}i}{20}. \quad (31)$$

Now let us construct rational solutions of equation (19). Substituting $w = z^m$ as z tends to infinity into equation (19), we see that $m = 0$. Note that at certain conditions on the parameters β, C_0, C_1, C_2 expansion (20) terminates. As a result we find a rational solution

$$w(z) = \frac{180}{z^4} + \frac{C_0}{6}, \quad \beta = 0, \quad C_1 = -\frac{C_0^2}{12}, \quad C_2 = \frac{C_0^3}{216}. \quad (32)$$

Elliptic and simply periodic solutions (26) (29) were given in [3–6]. Note that in the case $\beta = 0$ equation (19) does not have simply periodic solutions, while at certain conditions on the parameters C_0, C_1, C_0 this equation with $\beta = 0$ possesses elliptic and rational solutions.

4 Exact solutions of equation (1) with $n = 2$

In this section we construct meromorphic traveling wave solutions of the modified Kawahara equation (equation (1) with $n = 2$). Again we omit arbitrary constant z_0. The equation

$$w_{zzz}w_z - \frac{1}{2}w_{zz}^2 - \frac{\beta}{2}w_z^2 - 30w^4 + \frac{C_0}{2}w^2 + C_1 w + C_2 = 0. \quad (33)$$

possesses two different formal Laurent series in a neighborhood the pole $z = 0$

$$w^{(1)}(z) = \frac{1}{z^2} - \frac{\beta}{60} + \frac{10C_0 - \beta^2}{3600} z^2 + \frac{5C_0\beta + 450C_1 - \beta^3}{151200} z^4 + \dots,$$

$$(34)$$

$$w^{(2)}(z) = -\frac{1}{z^2} + \frac{\beta}{60} + \frac{\beta^2 - 10C_0}{3600} z^2 + \frac{\beta^3 + 450C_1 - 5C_0\beta}{151200} z^4 + \dots.$$

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The Fuchs indices for these expansions are the following -1, $(7 \pm \sqrt{71}i)/2$. Since none of them is a non–negative integer, we see that all the coefficients in expansions (34) are uniquely determined. First of all let us construct exact meromorphic solutions having poles of single type ($i = 1$ or $i = 2$). Note that equation (33) possesses the symmetry

$$w(z; -C_1) = -w(z; C_1) \tag{35}$$

Thus, without loss of generality, we need to find meromorphic solutions with poles of the first type. Necessary condition for existence of elliptic solutions is automatically satisfied. It follows from the theorem 2 that we should take an elliptic solution in the form

$$w(z) = c_{-2}^{(1)} \wp(z; g_2, g_3) + h_0, \quad c_{-2}^{(1)} = 1. \tag{36}$$

The Laurent series for this function in a neighborhood of the point $z = 0$ is the following

$$w(z) = \frac{c_{-2}^{(1)}}{z^2} + h_0 + \frac{c_{-2}^{(1)} g_2 z^2}{20} + \frac{c_{-2}^{(1)} g_3 z^4}{28} + o(|z|^4), \quad 0 < |z| < \tilde{\varepsilon}_1. \tag{37}$$

Comparing coefficients of this series with coefficients of the series $w^{(1)}(z)$ (see (34)), we find the parameters of elliptic solution (36)

$$h_0 = -\frac{\beta}{60}, \quad g_2 = \frac{10\,C_0 - \beta^2}{180}, \quad g_3 = \frac{5\,C_0\beta + 450\,C_1 - \beta^3}{5400} \tag{38}$$

and the correlation on the parameters of equation (33)

$$C_2 = \frac{32\,\beta^4 - 220\,C_0\beta^2 + 125\,C_0^2 - 8100\,C_1\beta}{324000}. \tag{39}$$

Substituting obtained elliptic solution and the correlation (39) into equation (33), we see that it indeed satisfies the equation provided that (39) holds. The explicit expression for simply periodic solution with the first type of poles is the following

$$w(z) = -\sqrt{L} c_{-2}^{(1)} \frac{d}{dz} \cot(\sqrt{L}z) + h_0 \equiv L c_{-2}^{(1)} [\cot^2(\sqrt{L}z) + 1] + h_0, \tag{40}$$

$$\sqrt{L} = \frac{\pi}{T}, \quad c_{-2}^{(1)} = 1.$$

The Laurent series for this function in a neighborhood of the point $z = 0$ can be written as

$$w(z) = \frac{c_{-2}^{(1)}}{z^2} + \frac{L c_{-2}^{(1)}}{3} + h_0 + \frac{L^2 c_{-2}^{(1)} z^2}{15} + \frac{2L^3 c_{-2}^{(1)} z^4}{189} + o(|z|^4), \quad 0 < |z| < \tilde{\varepsilon}_2. \tag{41}$$

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Comparing coefficients of this series with coefficients of the series $w^{(1)}(z)$ (see (34)), we determine the parameters of simply periodic solution (40)

$$L = \frac{\kappa_j}{60}, \quad h_0 = -\frac{\kappa_j + 3\beta}{180}, \quad \kappa_j = \pm\sqrt{15(10C_0 - \beta^2)}, \quad j = 1, 2 \quad (42)$$

and correlations on the parameters of equation (33)

$$C_1 = \frac{10\,\kappa_j\,C_0 - \kappa_j\,\beta^2 - 45\,\beta\,C_0 + 9\,\beta^3}{4050},$$

$$(43)$$

$$C_2 = \frac{14\,\beta^4 - 130\,\beta^2 C_0 + 125\,C_0^2 + 2\,\beta^3\kappa_j - 20\,\beta\,\kappa_j\,C_0}{324000}.$$

Checking–up obtained solutions, we see that equation (33) indeed possesses solutions of the form

$$w(z) = \frac{\kappa_j}{60}\cot^2\left(\frac{\sqrt{15\kappa_j}}{30}\,z\right) + \frac{2\kappa_j - 3\beta}{180}, \quad j = 1, 2, \quad (44)$$

if correlations (43) hold. In the same way we obtain the rational solution with the pole of the first type. The rational function

$$w(z) = \frac{1}{z^2} - \frac{\beta}{60}. \quad (45)$$

solves equation (33) provided that the following correlations hold

$$C_0 = \frac{\beta^2}{10}, \quad C_1 = \frac{\beta^3}{900}, \quad C_2 = \frac{\beta^4}{144000}. \quad (46)$$

Meromorphic solutions with poles of the second type may be found with the help of the symmetry (35). Now let us construct meromorphic solutions that posses poles of two types at the same type. Without loss of generality, let us suppose that the point $z = 0$ is a pole of the first type. The expression for elliptic solutions is the following

$$w(z) = c_{-2}^{(1)}\wp(z; g_2, g_3) + c_{-2}^{(2)}\wp(z - a; g_2, g_3) + \tilde{h}_0, \ c_{-2}^{(1)} = 1, \ c_{-2}^{(2)} = -1. \quad (47)$$

Expanding this function in a neighborhood of the points $z = 0$, $z = a$ and introducing notation $A \overset{def}{=} \wp(A)$, $B \overset{def}{=} \wp_z(a)$, yields

$$w(z) = \frac{c_{-2}^{(1)}}{z^2} + c_{-2}^{(2)}A + \tilde{h}_0 - c_{-2}^{(2)}Bz + o(|z|), \quad 0 < |z| < \tilde{\varepsilon}_3,$$

$$(48)$$

$$w(z) = \frac{c_{-2}^{(2)}}{(z - a)^2} + c_{-2}^{(1)}A + \tilde{h}_0 + c_{-2}^{(1)}(z - a) + o(|z - a|),$$

$$0 < |z - a| < \tilde{\varepsilon}_4.$$

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Note that using an addition formula for the Weierstrass elliptic function we can rewrite expression (47) as

$$w(z) = (c_{-2}^{(1)} - c_{-2}^{(2)}) \, \wp + \frac{c_{-2}^{(2)}}{4} \left(\frac{\wp_z + B}{\wp - A} \right)^2 + \tilde{h}_0 - c_{-2}^{(2)} A. \qquad (49)$$

Comparing coefficients of the series (48) with coefficients of expansions $w^{(1)}(z)$, $w^{(2)}(z)$ (see (34)), we obtain an algebraic system for the parameters of meromorphic solution (49). Solving this system added by equation $B^2 = 4A^3 - g_2 A - g_3$, we get

$$A = \frac{\beta}{60}, \; B = 0, \; \tilde{h}_0 = 0, \; g_2 = \frac{\beta^2 + 5 C_0}{540}, \; g_3 = -\frac{\beta \, (2 \, \beta^2 + 25 \, C_0)}{162000} \qquad (50)$$

Equation (33) possesses an elliptic solution of the form

$$w(z) = \frac{10800 \, \wp^2 - 360 \, \beta \, \wp + 25 \, C_0 - \beta^2}{180 \, (60 \, \wp - \beta)} \qquad (51)$$

provided that the following correlations hold

$$C_1 = 0, \quad C_2 = \frac{(11 \, \beta^2 + 100 \, C_0) \, (4 \, \beta^2 - 25 \, C_0)}{1215000}. \qquad (52)$$

According to theorem 2 we take an expression for simply periodic solutions in the form

$$w(z) = L c_{-2}^{(1)} [\cot^2(\sqrt{L} z) + 1] + L c_{-2}^{(2)} [\cot^2(\sqrt{L}(z - a)) + 1] + h_0, \qquad (53)$$

where $c_{-2}^{(1)} = 1$, $c_{-2}^{(2)} = -1$, $\sqrt{L} = \pi/T$. Expanding this function in a neighborhood of the points $z = 0$, $z = a$ and introducing notation $A \overset{def}{=} \sqrt{L} \cot(\sqrt{L} a)$, we obtain

$$w(z) = \frac{c_{-2}^{(1)}}{z^2} + \frac{c_{-2}^{(1)} L}{3} + c_{-2}^{(2)} \left(L + A^2 \right) + h_0 + 2 \, c_{-2}^{(2)} A \left(L + A^2 \right) z$$
$$+ o(|z|), \quad 0 < |z| < \tilde{\varepsilon}_5,$$
$$(54)$$

$$w(z) = \frac{c_{-2}^{(2)}}{(z - a)^2} + \frac{c_{-2}^{(2)} L}{3} + c_{-2}^{(1)} \left(L + A^2 \right) + h_0 - 2 \, c_{-2}^{(1)} A \left(L + A^2 \right)$$
$$\times (z - a) + o(|z - a|), \quad 0 < |z - a| < \tilde{\varepsilon}_6.$$

Comparing coefficients of these series with coefficients of expansions $w^{(1)}(z)$, $w^{(2)}(z)$ (see (34)), we find the parameters of meromorphic solution (53)

$$L = \frac{\beta}{40}, \quad A = 0, \quad h_0 = 0 \qquad (55)$$

and conditions on the parameters of equation (33)

$$C_0 = -\frac{11\beta^2}{100}, \quad C_1 = 0, \quad C_2 = 0. \tag{56}$$

In the case $A = 0$ we get $a = \sqrt{10\beta}\pi/\beta$. Verification shows that the following function

$$w(z) = \frac{\beta \cot\left(\frac{\sqrt{10\beta}z}{10}\right)}{10 \sin\left(\frac{\sqrt{10\beta}z}{10}\right)}. \tag{57}$$

indeed solves equation (33) with the parameters C_0, C_1, C_2 given by (56). If we try to find rational solutions with two poles, we shall see that arising algebraic system is inconsistent. Thus we have found the whole set of non–constant meromorphic solutions for (33). Note that elliptic solutions (36), (51) degenerate if the following condition is valid $g_2^3 - 27g_3^2 = 0$.

Simply periodic solutions with poles of one type arise in [3, 5]. Elliptic solutions with poles of one type were given in [6]. Solutions with poles of different types seem to be new.

5 Exact solutions of equation (1) with $n = 4$

In this section our goal is to find meromorphic traveling wave solutions of the modified Kawahara equation (equation (1) with $n = 4$). Recall that traveling wave solutions of equation (1) with $n = 4$ and α, δ given by (4) satisfy the following third order ordinary differential equation

$$w_{zzz}w_z - \frac{1}{2}w_{zz}^2 - \frac{\beta}{2}w_z^2 - 4w^6 + \frac{C_0}{2}w^2 + C_1w + C_2 = 0. \tag{58}$$

This equation possesses the symmetries of the form

$$w(z; -C_1, C_2) = -w(z; C_1, C_2), \quad w(z; iC_1, -C_2) = iw(z; C_1, C_2). \tag{59}$$

Equation (58) admits four different formal Laurent expansions in a neighborhood of the pole $z = 0$. They are the following

$$w^{(1)}(z) = \frac{1}{z} - \frac{\beta}{60}z + \frac{15C_0 - \beta^2}{180}z^3 + \frac{C_1}{96}z^4 \dots,$$

$$w^{(2)}(z) = -\frac{1}{z} + \frac{\beta}{60}z + \frac{\beta^2 - 15C_0}{180}z^3 + \frac{C_1}{96}z^4 + \dots.$$

$$w^{(3)}(z) = \frac{i}{z} - \frac{i\beta}{60}z + \frac{i(15C_0 - \beta^2)}{180}z^3 + \frac{C_1}{96}z^4 + \dots,$$

$$w^{(4)}(z) = -\frac{i}{z} + \frac{i\beta}{60}z + \frac{i(\beta^2 - 15C_0)}{180}z^3 + \frac{C_1}{96}z^4 + \dots.$$

(60)

All not written out coefficients are uniquely determined since the Fuchs indices of these expansions are -1, $(5 \pm \sqrt{39i})/2$. Let us construct exact meromorphic solutions with poles of one type. Without loss of generality, we may consider solutions with poles of the first type. Elliptic functions possessing one pole of the first order in a parallelogram of periods do not exist. According to theorem 2 we take an expression for simply periodic solutions in the form

$$w(z) = \sqrt{L}c_{-1}^{(1)}\cot(\sqrt{L}z) + h_0, \quad \sqrt{L} = \frac{\pi}{T}, \quad c_{-1}^{(1)} = 1. \tag{61}$$

Following the procedure described in section 2 and taking into account the symmetries (59), we obtain

$$w(z) = \pm\frac{\sqrt{5\beta}\cot\left(\frac{\sqrt{5\beta}}{10}z\right)}{10}, \quad C_0 = \frac{3\beta^2}{50}, \quad C_1 = 0, \quad C_2 = \frac{\beta^3}{1000} \tag{62}$$

and

$$w(z) = \pm\frac{\sqrt{5\beta}i\cot\left(\frac{\sqrt{5\beta}}{10}z\right)}{10}, \quad C_0 = \frac{3\beta^2}{50}, \quad C_1 = 0, \quad C_2 = -\frac{\beta^3}{1000}. \tag{63}$$

Again we omit arbitrary constant z_0. Rational solutions with one pole can be written as

$$w(z) = \pm\frac{1}{z}, \quad w(z) = \pm\frac{i}{z}, \quad \beta = 0, \quad C_0 = 0, \quad C_1 = 0, \quad C_2 = 0. \tag{64}$$

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Now let us consider the case of meromorphic solutions with poles of two different types. We begin with poles of the first and the second type. It follows from theorem 2 that an elliptic solution should be taken in the form

$$w(z) = c_{-1}^{(1)} \zeta(z; g_2, g_3) + c_{-1}^{(2)} \zeta(z - a; g_2, g_3) + \tilde{h}_0, \quad c_{-1}^{(1)} = -c_{-1}^{(2)} = 1. \quad (65)$$

This function possesses two poles $z = 0$, $z = a$ in a parallelogram of periods. Expanding this function in a neighborhood of the points $z = 0$, $z = a$ and introducing notation $A \overset{def}{=} \wp(a)$, $B \overset{def}{=} \wp_z(a)$, $h_0 \overset{def}{=} \tilde{h}_0 - c_{-1}^{(2)} \zeta(a)$, we get

$$w(z) = \frac{c_{-1}^{(1)}}{z} + h_0 - c_{-1}^{(2)} A z + \frac{c_{-1}^{(2)} B}{2} z^2 + o(|z|), \quad 0 < |z| < \tilde{\tilde{\varepsilon}}_1,$$

$$(66)$$

$$w(z) = \frac{c_{-1}^{(2)}}{z - a} + h_0 - c_{-1}^{(1)} A(z - a) - \frac{c_{-1}^{(1)} B}{2} (z - a)^2 + o(|z - a|),$$

$$0 < |z - a| < \tilde{\tilde{\varepsilon}}_2.$$

In new variables elliptic solution (65) can be rewritten as

$$w(z) = \frac{c_{-1}^{(2)}(\wp_z + B)}{2(\wp - A)} + h_0. \quad (67)$$

Comparing coefficients of the series (66) with coefficients of expansions $w^{(1)}(z)$, $w^{(2)}(z)$ (see (60)), we obtain the parameters of elliptic solution (67)

$$A = -\frac{\beta}{60}, \quad B = 0, \quad h_0 = 0, \quad g_2 = \frac{\beta^2 - 10\, C_0}{120},$$

$$g_3 = \frac{(13\, \beta^2 - 150\, C_0)\, \beta}{108000}. \quad (68)$$

In addition several correlations on the parameters of equation (58) arise. The elliptic solution and the correlations on the parameters can be written as

$$w_{1,2}(z) = \frac{\wp_z}{2(\wp + \frac{\beta}{60})}, \quad C_1 = 0, \quad C_2 = \frac{(9\, \beta^2 - 100\, C_0)\, \beta}{3000}. \quad (69)$$

Using symmetries (59), we obtain another elliptic solution of equation (58)

$$w_{3,4}(z) = \frac{i\wp_z}{2(\wp + \frac{\beta}{60})}, \quad C_1 = 0, \quad C_2 = -\frac{(9\, \beta^2 - 100\, C_0)\, \beta}{3000}. \quad (70)$$

In the same way we find the simply periodic solution with poles of the first and the second type. General expression for such a solution (see theorem 2) reads as

$$w(z) = \sqrt{L} c_{-1}^{(1)} \cot(\sqrt{L} z) + \sqrt{L} c_{-1}^{(2)} \cot(\sqrt{L}(z - a)) + h_0, \quad \sqrt{L} = \frac{\pi}{T}. \quad (71)$$

Equation (58) indeed possesses a solution of the form. This solution is the following

$$w_{1,2}(z) = \frac{\sqrt{10\beta}}{10\sinh\left(\frac{\sqrt{10\beta}z}{10}\right)}, \quad C_0 = \frac{9\beta^2}{100}, \quad C_1 = 0, \quad C_2 = 0. \qquad (72)$$

Making use of symmetries (59) we get the simply periodic solution with poles of the third and the fourth type

$$w_{3,4}(z) = \frac{\sqrt{10\beta}i}{10\sinh\left(\frac{\sqrt{10\beta}z}{10}\right)}, \quad C_0 = \frac{9\beta^2}{100}, \quad C_1 = 0, \quad C_2 = 0. \qquad (73)$$

Algebraic systems for the parameters are inconsistent for all other simply periodic solutions with poles of two different types as well as simply periodic solutions with poles of three and four different types. The same is true in the case of rational solutions with two, three, and four poles. As far as elliptic solutions are concerned, necessary condition (13) does not allow existence of elliptic solutions with three poles in a parallelogram of periods. In the case of elliptic solutions with four poles in a parallelogram of periods the algebraic system for the parameters is also inconsistent. Meromorphic solutions of equation (58) obtained in this section seem to be new.

6 Conclusion

In this article we have studied traveling wave solutions of the Kawahara and the modified Kawahara equations. We have found all the families of meromorphic solutions for ordinary differential equations (19), (33), (58) describing the traveling wave solutions of the Kawahara equation and its generalizations. In addition we have described a powerful method for constructing exact meromorphic solutions (including elliptic, simply periodic and rational solutions) of autonomous nonlinear ordinary differential equations. Our method generalizes several methods with an a priori fixed expression for an exact solution. Our method allows one to find the whole set of exact meromorphic solutions for a wide class of autonomous nonlinear ordinary differential equations. Besides that our method may be useful if one needs to classify meromorphic solutions of an autonomous nonlinear ordinary differential equation. Indeed, our approach involves calculation of the period and amount of poles in a stripe of periods for simply periodic solutions and invariants g_2, g_3 of the Weierstrass elliptic function $\wp(z)$ and amount of poles in a parallelogram of periods for elliptic solutions.

7   Acknowledgements

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